On Incentives for Sustainable Investments

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Abstract

There is a trend among institutional investors to split their assets between index-managers and specialists. The specialist mandates are typically delegated to specialist asset managers, who are assumed to generate "alpha", take on large risks and whose remuneration is performance based.

In this paper, we will study how the optimal behavior of the specialist manager will depend on the remuneration structure.

Keywords Incentives, portfolio choice, sustainable investments, value function.

subject JEL classification G23

1 Introduction

This paper highlights the effects on asset manager behavior, due to changes in the incentive schemes.

Variable incentive schemes are increasingly an important part of the total remuneration of asset managers by investors. Specialized managers in private equity, hedge-funds, *etc* are also increasing their share of managed assets. The total compensation to the asset manager is often a fixed fee and an incentive fee based on the excess return (if positive) related to some index; both of these fees are then proportional to the size of the mandate.

Some of the quoted reasons for this trend are: alignment of interest between investors and asset managers; the best assets management companies can therefore attract the most skilled individuals; and it allows the investor to allocate a smaller portion of total assets to a few (idiosyncratic?) specialist managers keeping the residual assets with a cost-effective index manager. Specialist asset managers, either with respect to the type of assets they manage, or with respect to the investment process; are therefore often those managers with incentive schemes.

Outsourcing the management of assets, introduces a principal (manager) agent (investor) relationship. This paper has therefore its origin in the agency literature.

Our analysis of managerial behavior is in terms of optimal choices between two risky assets and the manager is assumed to be maximizing the expected utility of his income. Four different managerial remuneration schemes are considered, including such with high-water mark based bonuses and liquidation due to poor performance of the fund. The high-water mark and liquidation setting was considered from a fund value perspective using a continuous time stochastic differential equation model by Goetzmann, Ingersoll and Ross [3], and by Hodder and Jackwerth [4] for an expected utility maximizing manager that has the choice of a risky asset and a riskless one. As will be seen below some of our results bear resemblance to those of [4]. As opposed to Goetzmann, Ingersoll and Ross [3], and Hodder and Jackwerth [4], the lower barrier considered in our work is not a fixed one, but rather one that is a fraction of one of the risky assets in which the manager is assumed to invest. The issue of which benchmark to use is addressed by Admati and Pfleiderer in [1].

We analyze managerial behavior in the four different remuneration schemes for both single time period and multi period settings. The results in the single time period problems can be found as partial results in the multi period settings. For example, the optimal choice in a single time period high-water mark setting starting with a fund value of one is the same as the optimal choice when both the fund value and high-water mark are one in the last period in the multi period setting.

Our work also aims at proposing incentive structures for environmental or corporate social responsibility (CSR) oriented funds. These incentive structures may have high-water mark based bonuses, but their key feature is to yield extra managerial income if the environmentally friendly (CSR or green) asset beats the conventional (index or black) asset during a given time period.

It is also our wish to find contracts which force the manager to invest the full amount provided by the investor in the manager's own (green) strategy. This relates to the work of Dybvig, Farnsworth and Carpenter [2].

2 The model

Here we specify the model that governs the development of the fund value.

We assume that the capital to be handled by the manager is initially equal to 1. At time points $t-1$, for $t = 1, \dots, T$, the manager allocates capital according to

$$\lambda_{t-1}\mathrm{e}^{X_t} + (1-\lambda_{t-1})\mathrm{e}^{Z_t},$$

where e^{X_t} represents a "green" investment and e^{Z_t} represents some "conventional" or "black" investment. The choice λ_{t-1} is a real number between zero and one. Letting $\tilde{Y}_{t-1}$ be the fund value at the time of choice, the fund value at time t is

$$Y_t = \left(\lambda_{t-1}\mathrm{e}^{X_t} + (1-\lambda_{t-1})\mathrm{e}^{Z_t}\right)\tilde{Y}_{t-1}.$$

In accordance with the Black-Scholes model, we assume that for $t \in \{1, \dots, T\}$, X_t are i.i.d. Gaussian random variables with mean $\Delta(\mu - 0.5\sigma^2)$ and variance $\Delta\sigma^2$ and that Z_t for are i.i.d. Gaussian random variables with mean $\Delta(\nu - 0.5\tau^2)$

and variance $\Delta\tau^2$. Here Δ is a time-scaling factor. We also assume that X_t and Z_t are correlated with correlation parameter $\rho > 0$.

The manager is assumed to have skills allowing him to compose a portfolio with higher yield than the benchmark. The manager's portfolio is also assumed to be more risky than the benchmark. In all numerical runs below these features are reflected in that the green asset (represented by e^{X_t}) has higher volatility and slightly higher positive drift than the black asset (represented by e^{Z_t}), i.e. $\sigma > \tau$ and $\mu - 0.5\sigma^2 > \nu - 0.5\tau^2$. The restriction $\rho > 0$ is due to the fact that the "green" asset is a portfolio with some components equal to those of the "black" asset.

Gaussian random variables are chosen for transparency. This is no restriction to our approach and we could just as well have chosen to work with some other distributions.

The fund value is registered by the investor at the discrete time points

$$t \in \{1 \ldots, T\},$$

and at each time point the manager receives a fixed fraction, denoted α, of the fund value and if he performs better than the "conventional" asset he receives a fraction, denoted β, of the excess return. The contract may also be of high-water mark type where the manager receives the β fraction only if the fund value is recorded at an all time high. We denote the managers fee at time t by Φ_t. Letting $f(\lambda_{t-1}) = \lambda_{t-1}e^{X_t} + (1-\lambda_{t-1})e^{Z_t}$, the value of the fund Y_t is assumed to be evolving according to

$$Y_t = f(\lambda_{t-1})(Y_{t-1} - \Phi_{t-1}), \tag{2.1}$$

where Φ_{t-1} is the manager's reward at time $t-1$. Henceforth

$$\tilde{Y}_t = Y_t - \Phi_t. \tag{2.2}$$

The remuneration schemes considered are;

Setting I This is the simplest setting and aims at forcing green investments. Here the green investment has to be monitored at a higher value than the black investment in order for the manager to receive his bonus. If this requirement is fulfilled the manager is awarded an amount proportional to the difference in values of the green and black asset. In mathematical terms, the manager's fee is given by

$$
\begin{aligned}
&\Phi_t(\lambda_{t-1}) &(2.3)\\
&= \left(\alpha \Delta f(\lambda_{t-1}) + \beta \max\left\{\lambda_{t-1}\mathrm{e}^{X_t} + (1-\lambda_{t-1})\mathrm{e}^{Z_t} - \mathrm{e}^{Z_t}; 0\right\}\right) \tilde{Y}_{t-1}\\
&= \left(\alpha \Delta f(\lambda_{t-1}) + \beta \lambda_{t-1} \max\left\{\mathrm{e}^{X_t} - \mathrm{e}^{Z_t}; 0\right\}\right) \tilde{Y}_{t-1}.
\end{aligned}
$$

Setting II The second type of reward is a development of **Setting I** and includes a lower barrier. This barrier is a fraction of the black investment. In order for the manager to be able to earn the full fee, the value of the fund has to be above the lower barrier at all monitoring points up until the time of reward claim. If the value of the fund is below the barrier at a monitoring point the manager loses his mandate and will only be rewarded a "welfare" amount at the current and remaining monitoring points. Letting a be the welfare amount for one time period, I_t be the value of the black investment, δ be the fraction and $M_t = \max\left\{\mathrm{e}^{X_t} - \mathrm{e}^{Z_t}; 0\right\}$, we have in this case that

$$
\begin{aligned}
\Phi_t(\lambda_{t-1}) = &\mathbf{1}(k=1)\Delta a + \mathbf{1}(k=0)\Big(\Delta a \mathbf{1}(Y_t < \delta I_t)\\
&+ \mathbf{1}(Y_t \geq \delta I_t)\left[\Delta \alpha f(\lambda_{t-1}) + \beta \lambda_{t-1} M_t\right] \tilde{Y}_{t-1}\Big), \qquad (2.4)
\end{aligned}
$$

where $k = 1$ indicates that the manager has lost his mandate.

Setting III The third remuneration scheme considered gives the manager the β fraction only if the fund has reached a high-water mark (i.e. a new maximum relative to the maximum of fund values from previous monitoring points) at the time of reward claim. Letting $H_t = \max\{Y_t, \ldots, Y_1, 1\}$ be the high-water mark, we have in this setting that

$$
\Phi_t(\lambda_{t-1}) = \Big(\alpha \Delta f(\lambda_{t-1}) + \beta \lambda_{t-1} M_t \mathbf{1}(Y_t > H_{t-1})\Big) \tilde{Y}_{t-1}. \qquad (2.5)
$$

Setting IV The fourth setting utilizes a high-water mark bonus and a barrier that is a fraction of the conventional investment (or index). Just as in **Setting II** the manager is knocked out and earns only welfare amounts (at the monitoring point and in remaining time periods) if the fund value is below the barrier at some monitoring point. We have that

$$\Phi_t(\lambda_{t-1}) = \mathbf{1}(k=1)\Delta a + \mathbf{1}_{\{k=0\}}\Big(\mathbf{1}(Y_t < \delta I_t)\Delta a \\ + \mathbf{1}(Y_t \geq I_t\delta)\Big[\alpha\Delta f(\lambda_{t-1}) + \beta\lambda_{t-1}M_t\mathbf{1}(Y_t > H_{t-1})\Big]\tilde{Y}_{t-1}\Big). \tag{2.6}$$

2.1 Parameter choices

Unless otherwise noted, parameters used in numerical runs are specified in the tables below.

For the distributions of the stochastic variables driving the asset prices we set parameters according to

green asset expected return	green asset volatility	black asset expected return	black asset volatility	correlation between green & black	time scale
μ	σ	ν	τ	ρ	Δ
0.08	0.3	0.03	0.2	0.7	1.0

And for the remuneration schemes we set

fixed fee fraction	bonus fraction	risk aversion	lower barrier percentage	welfare amount	discount rate
α	β	γ	δ	a	r
0.02	0.2	2	0.8	0.001	0.015

3 Theoretical analysis

Here we briefly outline the methodology for solving the optimization problems at hand.

Given the constant relative risk aversion (CRRA) utility function

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma},$$

and letting r be the discount rate, the manager's aim is to maximize the expected sum of utilities of his fees.

$$\mathbf{E}\left[\sum_{t=1}^{T} \mathrm{e}^{-r\Delta t} U\left(\Phi_t(\lambda_{t-1})\right)\right] = \frac{1}{1-\gamma}\mathbf{E}\left[\sum_{t=1}^{T} \mathrm{e}^{-r\Delta t}\Phi_t^{1-\gamma}(\lambda_{t-1})\right],$$

with respect to $\lambda_0, \ldots, \lambda_{T-1}$. The maximization will be done using dynamic programming methods. We define the value function

$$V_t(s) = \sup_{\lambda_{t-1}} \frac{1}{1-\gamma}\mathbf{E}\left[\sum_{i=t}^{T} \mathrm{e}^{-r\Delta}\Phi_i^{1-\gamma}(\lambda_{t-1})\Big|\, S_{t-1} = s\right],$$

where S_{t-1} is the state of the system at time $t-1$. Results from the dynamic programming literature, e.g., Stokey, Lucas and Prescott [5], give that the value function V_t satisfies the Bellman equation

$$V_t(s) = \sup_{\lambda} \mathrm{e}^{-r\Delta}\mathbf{E}\left[\frac{\Phi_t^{1-\gamma}(\lambda)}{1-\gamma} + V_{t+1}(S_t(\lambda))\Big|\, S_{t-1} = s\right],$$

for $t \in \{1, \ldots, T\}$, and where $V_{T+1} \equiv 0$.

So, at time $t = T-1$ we are to find the number $\hat{\lambda}_{T-1}(s)$ such that

$$\hat{\lambda}_{T-1}(s) = \mathrm{argmax}_{\lambda}\mathrm{e}^{-r\Delta}\mathbf{E}\left[\frac{\Phi_T^{1-\gamma}(\lambda)}{1-\gamma}\Big|\, S_{T-1} = s\right].$$

The corresponding value function is

$$V_T(s) = \sup_{\lambda} \mathrm{e}^{-r\Delta}\mathbf{E}\left[\frac{\Phi_T^{1-\gamma}(\lambda)}{1-\gamma}\Big|\, S_{T-1} = s\right].$$

At time points $t = T-2, \ldots, 0$, the problem is to find the numbers $\hat{\lambda}_t$ such that

$$\hat{\lambda}_t(s) = \mathrm{argmax}_{\lambda}\mathrm{e}^{-r\Delta}\mathbf{E}\left[\frac{\Phi_{t+1}^{1-\gamma}(\lambda)}{1-\gamma} + V_{t+2}(S_{t+1}(\lambda))\Big|\, S_t = s\right].$$

The value function is given by

$$V_{t+1}(s) = \sup_{\lambda} \mathrm{e}^{-r\Delta}\mathbf{E}\left[\frac{\Phi_{t+1}^{1-\gamma}(\lambda)}{1-\gamma} + V_{t+2}(S_{t+1}(\lambda))\Big|\, S_t = s\right].$$

Proceeding backwards recursively, we find the array of optimal choices

$$\{\hat{\lambda}_0, \hat{\lambda}_1(s_1), \ldots, \hat{\lambda}_{T-1}(s_{T-1})\}.$$

To solve numerically for the optimal choices, a tree of the state space going into each time period is spanned and the value function and the optimal choice are computed at each node of the tree using the optimization and interpolation routines in matlab.

4 Single period problem

In the one period problem there is only one choice variable λ. Below we present our numerical results on how the optimal choice $\hat{\lambda}$ depends on the parameters α and β in the different settings.

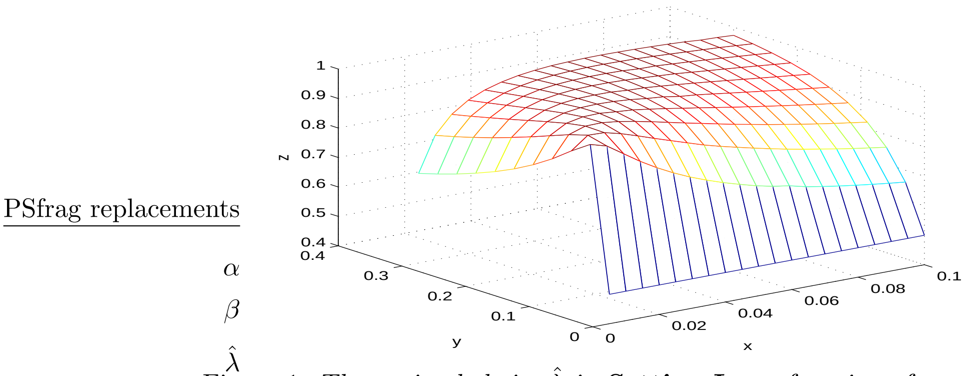


Figure 1: *The optimal choice $\hat{\lambda}$ in **Setting I** as a function of α and β.*

In **Setting I** the optimal choice is 0.5 if there is no bonus, i.e. if $\beta = 0$. We also note that there is a point close to the origin where the manager allocates all capital in the green asset. This indicates that it is not necessary for the investor to give the manager a large fixed fee or a large bonus in order to push green investments. Of course this is dependent on the parameters chosen for the random variables and may not hold for other parameter choices.

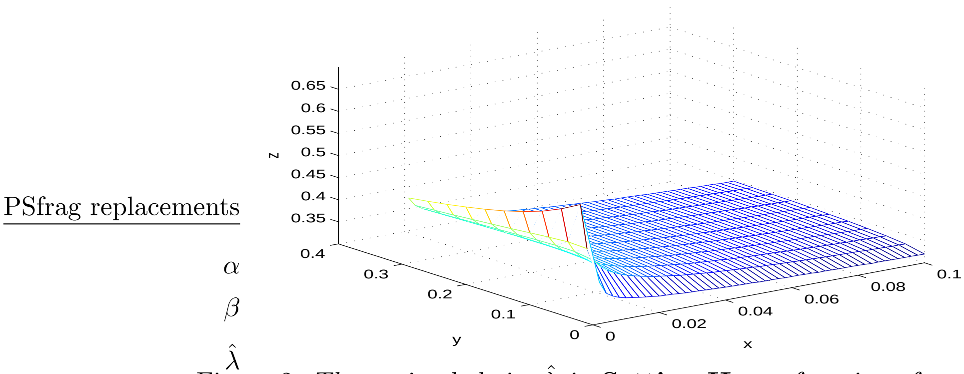


Figure 2: *The optimal choice $\hat{\lambda}$ in **Setting II** as a function of α and β.*

As one would expect, the introduction of the barrier yields overall less allocation in the more volatile asset. We also note that the optimal choice is now decreasing in α for $\beta = 0$. This means that, when no bonus is available, the manager is willing to take on more risk in order to earn a larger fixed fee. For **Setting I** and **Setting II** it is also interesting to see that for low values of α, i.e. small fixed fees, the optimal choice is decreasing in β. This has to do with the fact that the bonus part of the manager's fee is linear in β and λ. With a small fixed fee, a manager with CRRA utility is not willing to take on extra risk to earn a larger bonus. An increase in β is compensated by a decrease in λ which in turn gives a less risk for the fixed fee.

PSfrag replacements

α

β

$\hat{\lambda}$

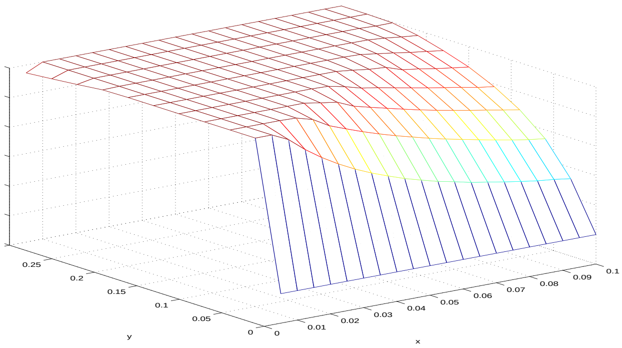


Figure 3: *The optimal choice $\hat{\lambda}$ as a function of α and β in* ***Setting III****.*

Just as in **Setting I** the optimal choice in **Setting III** is 0.5 if there is no bonus. Also here we note a slight roll off in optimal choice when α is small and β grows large.

PSfrag replacements

α

β

$\hat{\lambda}$

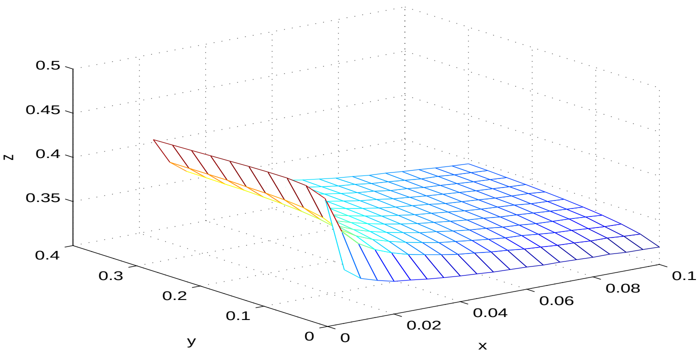


Figure 4: *The optimal choice $\hat{\lambda}$ as a function of α and β in* ***Setting IV****.*

Introducing the barrier in the high-water mark setting gives an even more drastic change in managerial behavior than adding the barrier to **Setting I**.

Just as in **Setting II** the optimal choice in **Setting IV** is decreasing in α for fixed β.

Simulations of 100000 runs with $\alpha = 0.02$ and $\beta = 0.2$ in the different settings gave the following means, 25%- and 75%-quantiles for the manager's fixed fee, bonus and total fee. For reference, we also give the optimal choices used.

Setting	I	II	III	IV
optimal choice	0.9615	0.3869	0.9999	0.3856
upper bound	0.0252	0.0238	0.0254	0.0238
fixed fee	0.0216	0.0210	0.0217	0.0210
lower bound	0.0170	0.0177	0.0169	0.0176
upper bound	0.0348	0.0057	0.0357	0.0104
bonus	0.0221	0.0036	0.0220	0.0075
lower bound	0.0000	0.0000	0.0000	0.0000
upper bound	0.0591	0.0286	0.0607	0.0343
total fee	0.0437	0.0247	0.0437	0.0285
lower bound	0.0170	0.0177	0.0169	0.0176

Clearly, earning bonuses in **SettingII** and **SettingIV**, i.e., the settings with barrier, is not an easy task. The high-water mark setting without barrier (**Setting III**), gives on average almost 500% higher bonuses than **Setting II**. Under our assumption of power utility $U(x) = x^{1-\gamma}/(1-\gamma)$ with $\gamma = 2$, the mean total fees in **SettingII** and **SettingIV** have to be increased by approx. 77% and 53%, respectively to give the same utilities as the mean total fees in **Setting I** and **Setting III**.

For the investor the fund value after rewarding the manager is of course of great interest. In the table below we give the mean and 95% confidence bounds in the four different settings.

Setting	I	II	III	IV
upper bound	1.0390	1.0280	1.0408	1.0233
fund mean	1.0378	1.0266	1.0390	1.0219
lower bound	1.0354	1.0252	1.0371	1.0205

Clearly, **Setting I** or **Setting III** is what the investor would prefer. According to our simulations, the introduction of a barrier is not feasible for the investor.

5 Multi-period problem

5.1 Setting I

This is the simplest setting considered where at each monitoring point the manager is awarded a fixed fraction of the of the fund value and another fraction of the fund value given that the green investment is monitored at a higher value than the black investment. In mathematical terms, the fee Φ_t at time t can written

$$\Phi_t(\lambda_{t-1}) = (\alpha\Delta f(\lambda_{t-1}) + \beta\lambda_{t-1}M_t)\,\tilde{Y}_{t-1}.$$

For notational convenience we let

$$g(\lambda_{t-1}) = \alpha\Delta f(\lambda_{t-1}) + \lambda_{t-1}\beta M_t. \tag{5.7}$$

At time $t = T-1$ the optimal choice is

$$\begin{aligned}\hat{\lambda}_{T-1}(y) &= \mathrm{argmax}_\lambda \frac{\mathrm{e}^{-\Delta r}}{1-\gamma}\mathbf{E}\left[\left(g(\lambda)\tilde{Y}_{T-1}\right)^{1-\gamma}\middle|\tilde{Y}_{T-1}=y\right] \\ &= \mathrm{argmax}_\lambda \frac{\mathrm{e}^{-\Delta r}y^{1-\gamma}}{1-\gamma}\mathbf{E}\left[g^{1-\gamma}(\lambda)\right],\end{aligned} \tag{5.8}$$

where y is the fund value at the time of choice, i.e. instantly after the manager is rewarded the amount Φ_{T-1}. Due to independence of the $g(\lambda)$ and $\tilde{Y}_{T-1}$, the optimal choice is independent of the fund value y. The optimal choice $\hat{\lambda}_{T-1}$ is found using the "fminbound" routine in matlab. The value function V_T is the right hand side of (5.8) with the argmax_λ replaced by $\sup_\lambda$. At time $t = T-2$, the optimal choice is

$$\hat{\lambda}_{T-2}(y) = \text{argmax}_\lambda \mathrm{e}^{-\Delta r}\mathbf{E}\left[\frac{\Phi_{T-1}^{1-\gamma}(\lambda)}{1-\gamma} + V_T(\tilde{Y}_{T-1}(\lambda))\middle|\tilde{Y}_{T-2} = y\right],$$

where $\tilde{Y}_{T-1}(\lambda)$ is the fund value at time $t = T-1$ given that choice and fund value at $t = T-2$ are λ and y, respectively. Using equations (2.2), (5.7) and (5.8), we have, for $t \in \{1, \dots, T\}$, that

$$\tilde{Y}_t(\lambda_{t-1}) = (f(\lambda_{t-1}) - g(\lambda_{t-1}))\tilde{Y}_{t-1},$$

so that

$$\begin{aligned}
&\mathbf{E}\left[\frac{\Phi_{T-1}^{1-\gamma}(\lambda)}{1-\gamma} + V_T(\tilde{Y}_{T-1}(\lambda))\middle|\tilde{Y}_{T-2} = y\right] \\
&= \frac{y^{1-\gamma}}{1-\gamma}\mathbf{E}\Big[g(\lambda)^{1-\gamma} + \mathrm{e}^{-\Delta r}\left([f(\lambda) - g(\lambda)]\, g(\hat{\lambda}_{T-1})\right)^{1-\gamma}\Big].
\end{aligned}$$

Again, the optimal choice again is independent of the current fund value. Proceeding recursively backwards we find the remaining optimal choices $\hat{\lambda}_0, \dots, \hat{\lambda}_{T-3}$. The backward recursion also yields that all optimal choices are independent of the fund values.

For some specific choices of α and β the optimal choices over four time periods are

α	β	$\hat{\lambda}_0$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$
	0	0.500	0.500	0.500	0.500
0.01	0.1	0.721	0.769	0.839	0.962
	0.2	0.672	0.717	0.780	0.884
	0	0.500	0.500	0.500	0.500
0.02	0.1	0.706	0.759	0.840	0.999
	0.2	0.690	0.745	0.825	0.962
	0	0.500	0.500	0.500	0.500
0.03	0.1	0.679	0.733	0.817	0.977
	0.2	0.682	0.741	0.830	0.985

We note that with parameters as chosen here the optimal choices are increasing over time unless the bonus option is removed (i.e. unless $\beta = 0$).

5.2 Setting II

This setting introduces a barrier that is a fraction of the conventional investment or index. If the fund value goes below some fraction of the index, the manager loses his mandate and is considered "out". Letting Ω_t be the "in-out" variable, where $\Omega_t = 0$ means "in" and $\Omega_t = 1$ "out", I_t be the value of the conventional investment or index and δ be the fraction, the optimal choice at time $t = T-1$ is

$$
\begin{aligned}
&\hat{\lambda}_{T-1}(i, y, k) \\
&= \text{argmax}_\lambda \frac{\mathrm{e}^{-\Delta r}}{1-\gamma} \mathbf{E}\left[\Phi_T^{1-\gamma}(\lambda) \middle| I_{T-1} = i, \tilde{Y}_{T-1} = y, \Omega_{T-1} = k \right] \\
&= \text{argmax}_\lambda \frac{\mathrm{e}^{-\Delta r}}{1-\gamma} \Big(\mathbf{1}(k=1)\Delta a + \mathbf{1}(k=0)\mathbf{E}\Big[\mathbf{1}\left(f(\lambda) y < \mathrm{e}^{Z_T} i\delta \right) \Delta a \\
&\qquad\qquad\qquad + \mathbf{1}\left(f(\lambda) y \geq \mathrm{e}^{Z_T} i\delta \right) \left(\Delta\alpha f(\lambda) + \beta\lambda M_T \right) y \Big] \Big)^{1-\gamma}.
\end{aligned}
$$

The value function V_T is given by replacing argmax_λ by $\sup_\lambda$. At time $t = T-2$ the optimal choice, given that $I_{T-2} = i$, $\tilde{Y}_{T-2} = y$ and $\Omega_{T-2} = k$ is

$$
\begin{aligned}
&\hat{\lambda}_{T-2}(i, y, k) = \text{argmax}_\lambda \frac{\mathrm{e}^{-\Delta r}}{1-\gamma} \\
&\times \mathbf{E}\left[\Phi_{T-1}^{1-\gamma}(\lambda) + V_T(I_{T-1}, \tilde{Y}_{T-1}(\lambda), \Omega_{T-1}(\lambda)) \middle| I_{T-2} = i, \tilde{Y}_{T-2} = y, \Omega_{T-2} = k \right],
\end{aligned}
$$

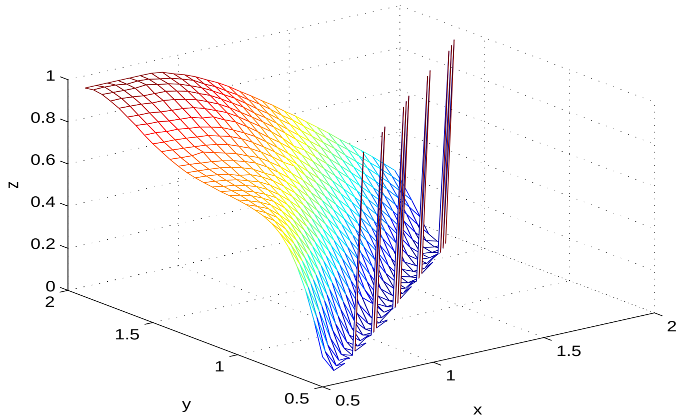
where I_{T-1}, $\tilde{Y}_{T-1}(\lambda)$ and $\Omega_{T-1}(\lambda)$ are the index value, fund value and in-out value at time $t = T-1$ given that the choice at time $t = T-2$ is λ and given that $I_{T-2} = i$, $\tilde{Y}_{T-2} = y$ and $\Omega_{T-2} = k$.

Proceeding backwards recursively we find the optimal choices and value functions for the remaining monitoring time points.

For a four period setting, the optimal choices are displayed below.

The optimal starting choice is

$$\hat{\lambda}_0 = 0.1232.$$



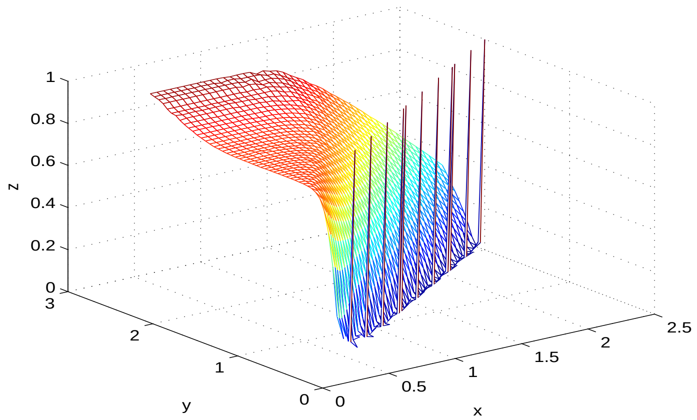
PSfrag replacements

index

fund

choice

Figure 5: *The optimal choice* $\hat{\lambda}_1$ *in* ***Setting II***



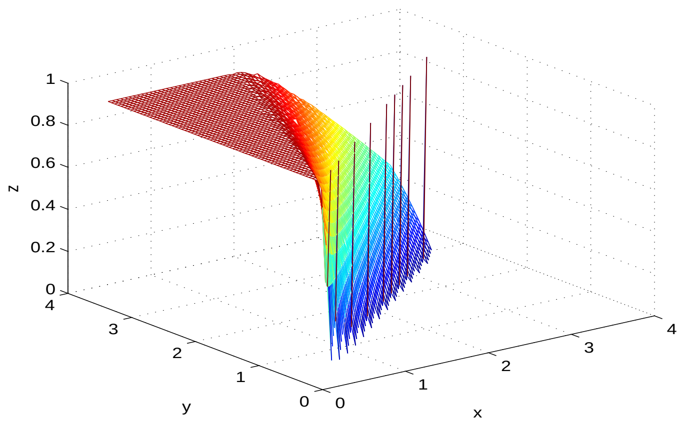
PSfrag replacements

index

fund

choice

Figure 6: *The optimal choice* $\hat{\lambda}_2$ *in* ***Setting II***



PSfrag replacements

index

fund

choice

Figure 7: *The optimal choice* $\hat{\lambda}_3$ *in* ***Setting II***

For choices $\hat{\lambda}_1$ and $\hat{\lambda}_2$ we note that for fund values high above the index value the proportion invested in the riskier (green) asset is close to one. This behavior is reflected by the fact that the bonus is linear in the choice variable. As the fund value approaches the barrier, which is 80% of the index value, the risk taking decreases. But very close to the barrier, when the manager is close to losing his mandate, a "when in trouble double" type of behavior is displayed.

Below are histograms of the optimal choices for 100000 simulated runs. From left to right we see $\hat{\lambda}_1$, $\hat{\lambda}_2$ and $\hat{\lambda}_3$.

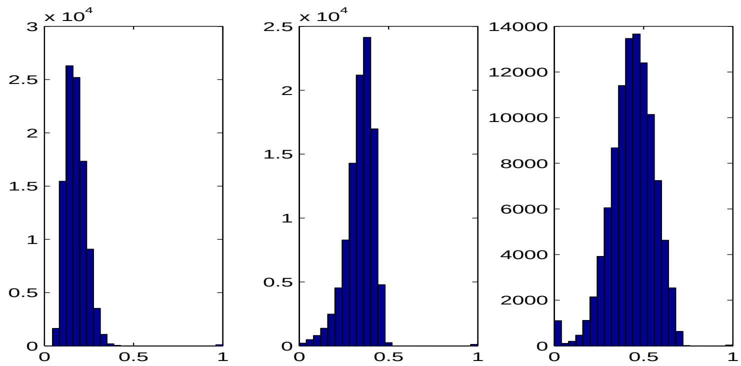


Figure 8: *The optimal choices from 100000 simulations in* ***Setting II****.*

It should be noted that the choice is zero only when the manager loses his mandate. From the histogram to the right, we see that the manager loses his mandate in roughly one percent of the runs. Compared to **Setting I** with $\alpha = 0.02$ and $\beta = 0.2$, we note that allocations in the more volatile green asset are less on average. We also see that allocation in the green asset is increasing on average.

5.3 Setting III

Here the manager receives a fixed fraction of the fund value and a bonus fraction given that the fund value at the monitoring point exceeds the fund values at all previous monitoring points, i.e. the fund value reaches a high-water mark. This means that the optimal choices are dependent on the current values of the fund and high-water mark, denoted y and h, respectively. It should be noted that the high-water mark is recorded *before* rewarding the manager and that the fund value is recorded *after* rewarding the manager. With $S_t = (\tilde{Y}_t, H_t)$.

and with Φ_t given by (2.5), the Bellman equation becomes

$$V_t(y,h) = \sup_{\lambda} \mathrm{e}^{-\Delta r} \mathbf{E}\left[U(\Phi_t(\lambda)) + V_{t+1}(\tilde{Y}_t(\lambda), H_t) \middle| \tilde{Y}_{t-1} = y, H_{t-1} = h \right]. \quad (5.9)$$

We note that

$$\begin{aligned} &\mathbf{E}\left[U(\Phi_t(\lambda)) \middle| \tilde{Y}_{t-1} = y, H_{t-1} = h \right] \\ &= \frac{1}{1-\gamma} \int_{\mathbb{R}^2} \Big(\alpha \Delta \tilde{f}(\lambda) y + \lambda \beta \max\{\mathrm{e}^x - \mathrm{e}^z; 0\} \mathbf{1}(\tilde{f}(\lambda) y > h) \Big)^{1-\gamma} f_{XZ}(x,z) dx dz \end{aligned}$$

and

$$\begin{aligned} &\mathbf{E}\left[V_{t+1}(\tilde{Y}_t(\lambda), H_t(\lambda)) \middle| \tilde{Y}_{t-1} = y, H_{t-1} = h \right] \\ &= \int_{\mathbb{R}^2} V_{t+1}\Big((1-\alpha\Delta) \tilde{f}(\lambda) y - \lambda \beta \max\{\mathrm{e}^x - \mathrm{e}^z; 0\} \mathbf{1}(\tilde{f}(\lambda) y > h), \\ &\qquad \tilde{f}(\lambda) y \mathbf{1}(\tilde{f}(\lambda) y > h) + h \mathbf{1}(\tilde{f}(\lambda) y \le h) \Big) f_{XZ}(x,z) dx dz, \end{aligned}$$

where $\tilde{f}(\lambda) = \lambda \mathrm{e}^x + (1-\lambda)\mathrm{e}^z$ and f_{XZ} is the joint density of X_t and Z_t. The optimal choices $\hat{\lambda}_{t-1}$, for $t = 1, \ldots, T$, are given upon replacing the sup on the right hand side of (5.9) by argmax.

Below are the optimal choices in **Setting III** for four periods. The optimal starting choice is

$$\hat{\lambda}_0 = 0.999.$$

The the second, third and fourth optimal choices as functions of the fund value and the high-water mark are found in the graphs below.

PSfrag replacements

fund

hwm

choice

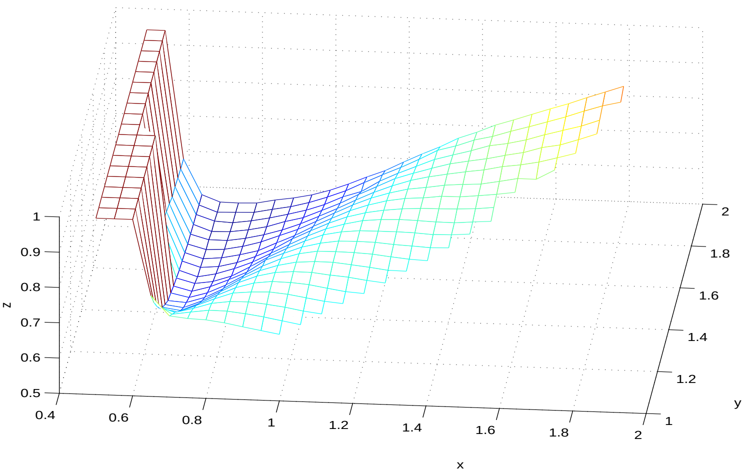


Figure 9: *The optimal choice $\hat{\lambda}_1$ in* ***Setting III*** *after the first period as a function of the fund value and the high-water mark.*

In the $\hat{\lambda}_1$ plot we see that for low fund values the manager optimally "goes all in" in the green asset. We also note a ridge where the fund value is slightly below the high-water mark. This implies that in these cases the manager optimally takes on a little more risk in order to gain a bonus.

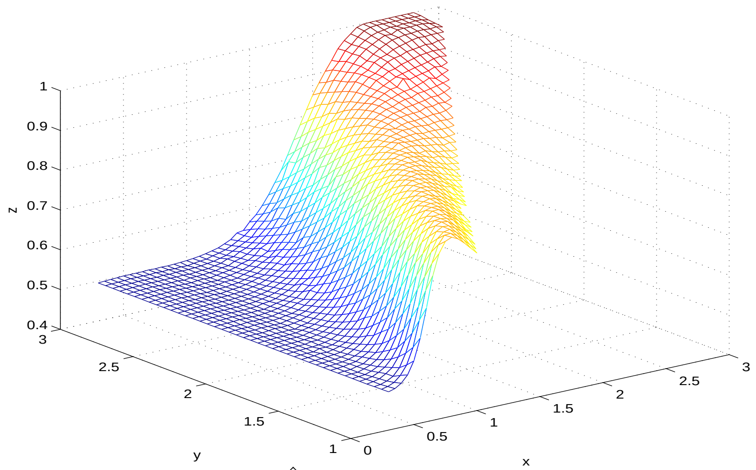


Figure 10: *The optimal choice* $\hat{\lambda}_2$ *in* ***Setting III*** *after the second period as a function of the fund value and the high-water mark.*

The $\hat{\lambda}_2$ plot bears similarities with the $\hat{\lambda}_1$ plot but here we also see slightly lowered risk tasking when the fund value is very low. This is due to the fact that earning a large fixed amount here will decrease the fund value to an extent such that the sum of the fixed fee earned here and the fixed fee earned in the last period will be less than it optimally could be.

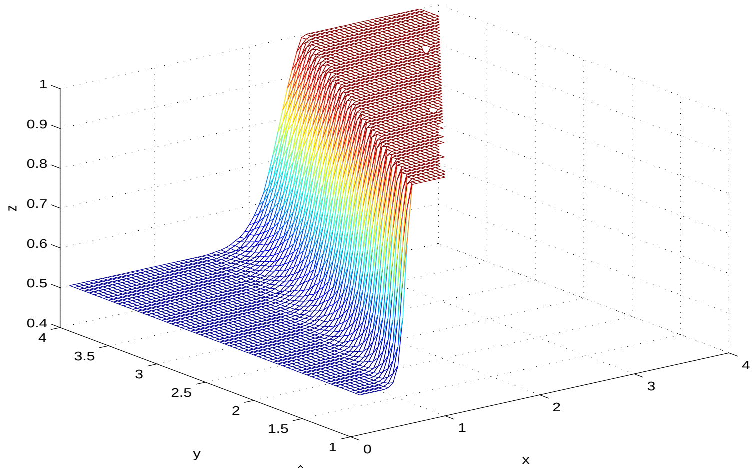


Figure 11: *The optimal choice* $\hat{\lambda}_3$ *in* ***Setting III*** *after the third period as a function of the fund value and the high-water mark.*

We note some quite drastic differences in the plots of optimal choices. Using the terminology of Hodder and Jackwerth [4] there is an "option ridge" present in the plots for $\hat{\lambda}_1$, $\hat{\lambda}_2$, but not for $\hat{\lambda}_3$. This is in accordance with the findings of Hodder and Jackwerth [4] for the case where the "black" asset or index is replaced by a riskless one. In the $\hat{\lambda}_2$ and $\hat{\lambda}_3$ plots we also see the "Merton flats" for high high-water marks and low fund values, where the manager invests as if the bonus is unattainable. At the the straight line where the fund value equals the high-water mark the optimal choice exactly the as in the one period problem.

Below are histograms of the optimal choices for 100000 runs. From left to right we see $\hat{\lambda}_1$, $\hat{\lambda}_2$ and $\hat{\lambda}_3$.

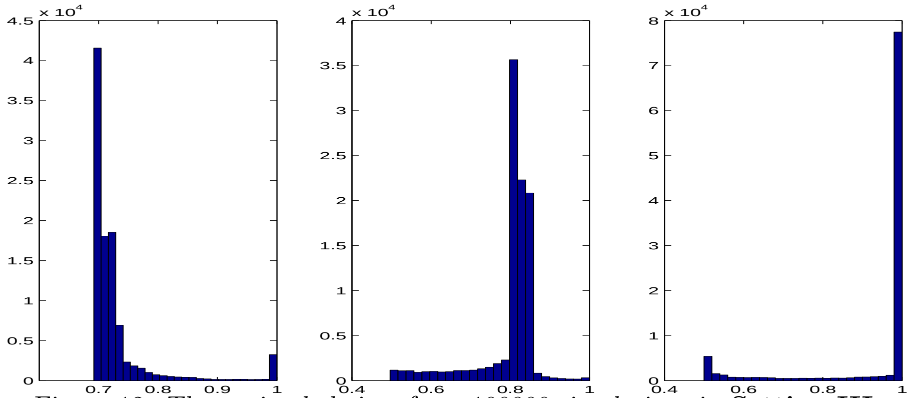


Figure 12: *The optimal choices from 100000 simulations in* ***Setting III***.

Going chronologically through time periods, we see that the choices are increasing on average. This means that the manager optimally takes on more risk as he approaches termination of his contract. We also note that even though the Merton flat is rather large in figure 11, trajectories only end up here in roughly five percent of the runs, whereas the optimal choice is one or very close to one in a little less than 80 percent of the runs.

5.4 Setting IV

Here a high-water mark bonus is combined with a barrier that is a fraction of the conventional investment or index. Letting $S_t = (I_t, H_t, \tilde{Y}_t, \Omega_t)$ and $s = (i, h, y, k)$, the optimal choice at time $t = T - 1$ is

$$
\begin{aligned}
\hat{\lambda}_{T-1}(s) &= \mathrm{argmax}_{\lambda} \frac{\mathrm{e}^{-\Delta r}}{1-\gamma} \mathbf{E}\left[\Phi_T^{1-\gamma}(\lambda) \middle| S_{T-1} = s \right] \\
&= \mathrm{argmax}_{\lambda} \frac{\mathrm{e}^{-\Delta r}}{1-\gamma} \Big(\mathbf{1}(k=1)\Delta a + \mathbf{1}(k=0)\mathbf{E}\Big[\mathbf{1}\left(f(\lambda)y < \mathrm{e}^{Z_T} i\delta\right)\Delta a \\
&\qquad + \mathbf{1}\left(f(\lambda)y \geq \mathrm{e}^{Z_T} i\delta\right)\left(\Delta\alpha f(\lambda) + \beta\lambda M_T \mathbf{1}(f(\lambda)y > h)\right) y \Big]\Big)^{1-\gamma}.
\end{aligned}
$$

The value function $V_T(s)$ is given by replacing argmax_λ by $\sup_\lambda$. The optimal choices at $t-1 = 0, \ldots, T-2$, given that $S_{t-1} = s$ are

$$
\hat{\lambda}_{t-1}(s) = \mathrm{argmax}_{\lambda} \frac{\mathrm{e}^{-\Delta r}}{1-\gamma} \mathbf{E}\left[\Phi_t^{1-\gamma}(\lambda) + V_{t+1}(S_t(\lambda)) \middle| S_{t-1} = s \right],
$$

where $S_t(\lambda) = (I_t, H_t(\lambda), \tilde{Y}_t(\lambda), \Omega_t(\lambda))$ are the index value, high-water mark, fund value and in-out value at time t given that the choice at time $t-1$ is λ and given that $I_{t-1} = i$, $H_{t-1} = h$, $\tilde{Y}_{t-1} = y$ and $\Omega_{t-1} = k$.

The numerical results, in a four period setting, are as follows; The initial optimal choice is

$$
\hat{\lambda}_0 = 0.064.
$$

The second, third and fourth optimal choices as a functions of the index and fund value and with the high-water mark fixed is given in the graphs below.

PSfrag replacements

index

fund

choice

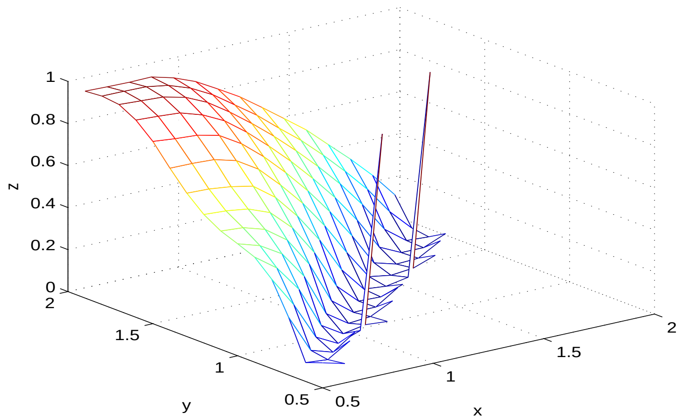


Figure 13: *The optimal choice $\hat{\lambda}_1$ in **Setting IV** after the first period as a function of the fund value and the index with the high-water mark fixed at* 2.1.

PSfrag replacements

index

fund

choice

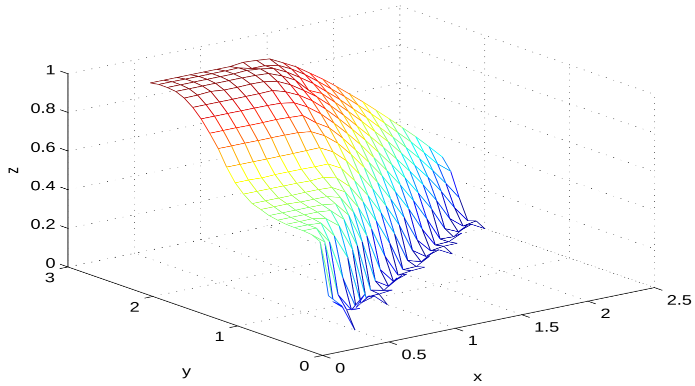


Figure 14: *The optimal choice* $\hat{\lambda}_2$ *in* ***Setting IV*** *after the second period as a function of the fund value and the index with the high-water mark fixed at* 2.7.

In figures 13 and 14 we see some indications of the ”when in trouble double behavior” that is also seen in **Setting II**. This behavior will become more obvious with finer numerical resolution. The option ridge, which also can be seen in figures 9 and 10 and in the work of Hodder of Jackworth [4], is present. The intuition is that for fund values slightly below the high-water mark the manager is willing to allocate all capital in the riskier green asset to have a chance of earning a bonus. We see Merton flats in figures 14 and 15. These occur when there is no obvious chance of earning a bonus and this is the case for fund values that are sufficiently smaller than the high-water mark but not so small that there is a risk for the manager of losing his mandate.

PSfrag replacements

index

fund

choice

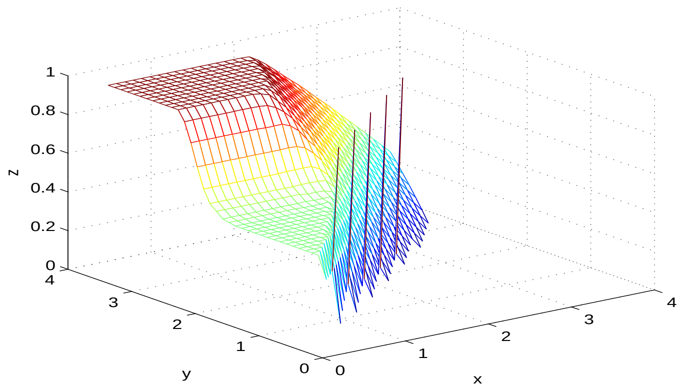


Figure 15: *The optimal choice* $\hat{\lambda}_3$ *in* ***Setting IV*** *after the third period as a function of the fund value and the index with the high-water mark fixed at* 3.7.

Below are histograms of the optimal choices for 100000 runs. From left to right we see $\hat{\lambda}_1$, $\hat{\lambda}_2$ and $\hat{\lambda}_3$.

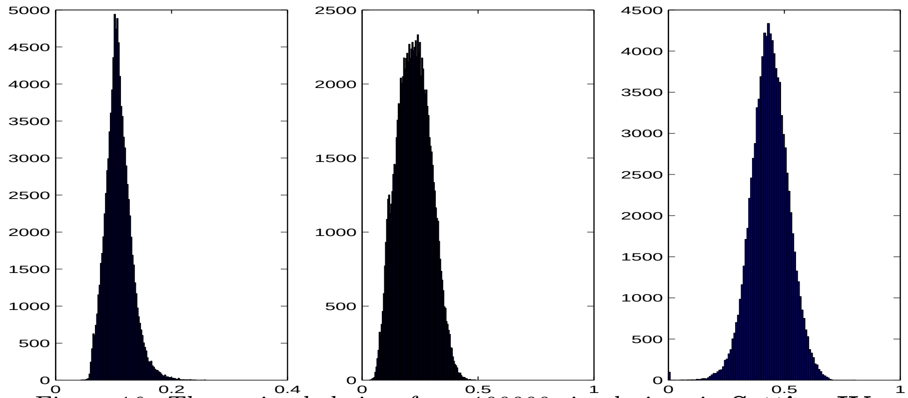


Figure 16: *The optimal choices from 100000 simulations in **Setting IV**.*

As in **Setting II**, the presence of the barrier gives less allocation in the more volatile green asset compared to the more risk friendly incentive structures in **Setting I** and **Setting III**. We also note that allocation in the green asset is increasing on average.

5.5 Comparison of the four different settings

In this section we compare the performance of the manager between the four settings. We also look at the differences in terminal fund values, which is interesting from the investor's point of view.

Below we display the means, 25%- and 75%-quantiles for the manager's fixed fees, bonuses and total fees, at time points $t = 1, 2, 3, 4$, in the four different settings.

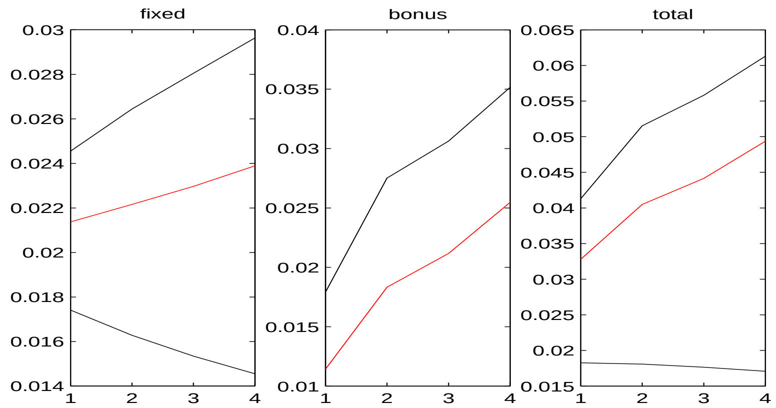


Figure 17: *Fees in* ***Setting I****.*

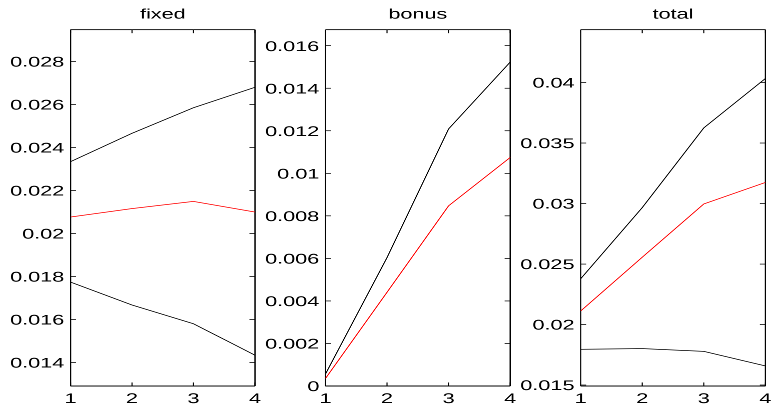


Figure 18: *Fees in* ***Setting II****.*

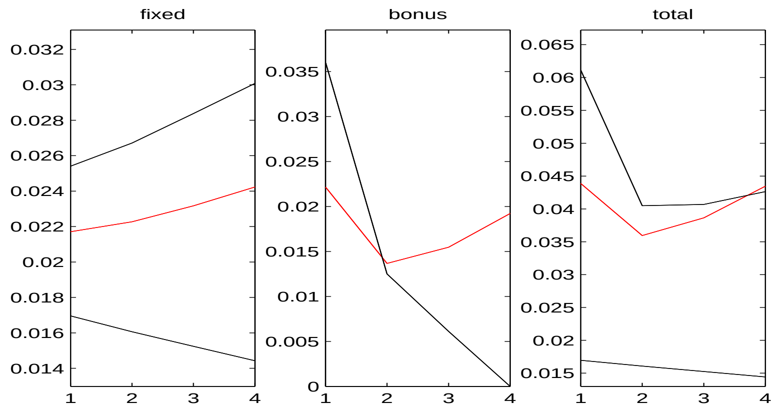


Figure 19: *Fees in* ***Setting III****.*

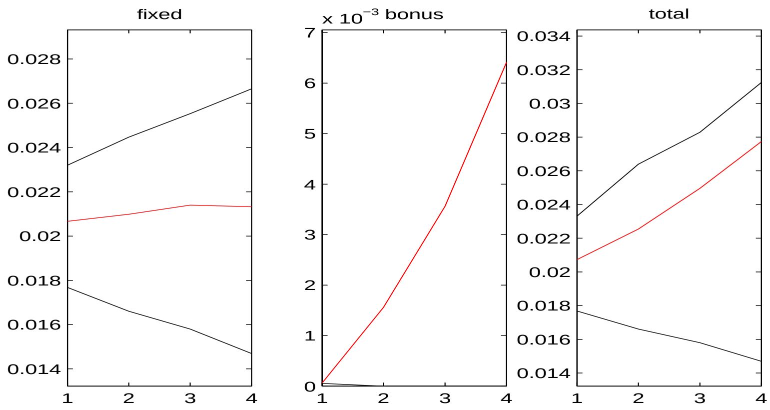


Figure 20: *Fees in **Setting IV**.*

Clearly, **Setting I** or **Setting III** is what the manager would prefer as these settings yield the highest rewards on average. We note that in **Setting III** and **Setting IV**, i.e. the settings with high-water mark bonus, the 75%-quantiles are below the means. This indicates that there are some substantial bonuses paid to the manager in runs where the fund values increase drastically. In **Setting III** and **Setting IV**, we also see that more than 75 percent of the runs yield no bonus in the last time period.

From the investor's point of view, the terminal fund values are of course of most interest. In the below table we give the mean, 25%- and 75%-quantiles for the fund values at termination.

Setting	I	II	III	IV
upper bound	1.4187	1.3124	1.4469	1.3047
mean	1.1452	1.0772	1.1683	1.0760
lower bound	0.7050	0.7404	0.7067	0.7473

As seen above the introduction of a lower barrier induces less allocation in the riskier green asset and this is reflected here in that the spread between upper and lower bounds for **Setting II** and **Setting IV** are narrower than those for **Setting I** and **Setting III**. Also the barrier-less settings have higher terminal fund value means than the ones with barrier, so from the investor's perspective a barrier is not necessarily a good thing even though it may reduce losses.

Also, since the manager's main task is to beat the index we the fund values relative to the index in the table below.

Setting	I	II	III	IV
upper bound	1.1858	1.0276	1.2281	1.0184
mean	1.0057	0.9517	1.0298	0.9530
lower bound	0.7749	0.8771	0.7670	0.8852

From this table it is clear that the introduction of a barrier is on average not feasible at all for the investor. Also the only incentive scheme that seems to be able to generate substantial wealth for the investor is **Setting III**.

6 Conclusions

We have seen that different features in portfolio managers' remuneration schemes lead to quite drastic differences in optimal choices between two risky assets. Especially, we have noted that the introduction of a barrier decreases risk, but on the other hand it gives less opportunity for the manager *and* investor to maximize their profits. We have also seen that the barrier gives birth to a desperate type of behavior when the fund value is very close to the barrier. From a corporate social responsibility point of view, we have proposed a simple incentive structure (**Setting I**) that may yield substantial income at least for the manager. Combined with a high water mark, i.e. **Setting III**, also the investor has a fair chance of earning substantial amounts. We have also found that some features present in the high-water mark setting with a risky asset and a riskless one, as in the work of Hodder and Jackwerth [4], are still present when the riskless asset is replaced by another risky asset.

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References

[1] Admati, A.R.; and Pfleiderer, P. "Does it All Add Up? Benchmarks and the Compensation of Active Portfolio Managers". *The Journal of Business,* Vol. 70, **3** (1997), pp. 323-350.

[2] Dybvig P.H.; Farnsworth H.K.; and Carpenter, J.N. "Portfolio Performance and Agency" *The Review of Financial Studies,* http://rfs.oxfordjournals.org/cgi/content/abstract/hhp056.

[3] Goetzmann, W.N.; Ingersoll, J.E. Jr.; and Ross, S.A. "High-Water Marks and Hedge Fund Management Contracts". *Journal of Finance,* 58 (2003), 1685-1717.

[4] Hodder, J.E.; and Jackwerth, J.C. "Incentive Contracts and Hedge Fund Management". *Journal of Financial and Quantitative Analysis,* 42 **4** (2007), 811-826.

[5] Stokey, N.; Lucas, R.E.; and Prescott, E. (1989) *Recursive Methods in Economic Dynamics* Harvard University Press.